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## Master's Thesis

# The Construction and Numerical Implementation of an Adaptive Schemes for the elasticity problems

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# Contents

# 1 Introduction

The construction and numerical implementation of adaptive schemes for elasticity problems have been a subject of significant interest and research. In this master report, we present the development and application of adaptive schemes that combine the finite element method (FEM) and boundary element method (BEM) to address elasticity problems.

Elasticity problems involve the analysis and understanding of the deformation and stress distribution in solid materials under external forces. The accurate and efficient solution of such problems is crucial for a wide range of engineering applications, including structural design, material characterization, and failure prediction.

Traditional numerical methods, such as FEM and BEM, have been widely used to solve elasticity problems individually. The FEM discretizes the problem domain into finite elements, allowing for the approximation of the solution within each element. On the other hand, the BEM represents the problem domain's boundary using surface elements, enabling the direct computation of the solution on the boundary.

The combination of FEM and BEM, known as the FEM-BEM coupling or hybrid methods, offers several advantages. By leveraging the strengths of both methods, it is possible to achieve improved accuracy, reduced computational effort, and more efficient handling of complex geometries compared to using either method alone. Adaptive schemes within this framework further enhance the numerical solution by adapting the mesh and element size to the solution's local features.

In this report, we focus on the construction and numerical implementation of adaptive schemes for elasticity problems. Our approach involves dynamically adjusting the mesh and element size based on local error indicators and solution features. This adaptivity enables the concentration of computational resources in regions where accuracy is crucial while reducing computational effort in areas of less significance.

The developed adaptive schemes aim to provide reliable and efficient solutions for elasticity problems encountered in engineering practice. We present the theoretical foundations of the combined FEM-BEM approach, outline the construction and implementation of the adaptive schemes, and demonstrate their effectiveness through numerical examples.

# 2 Formulation of the elasticity problem

## 2.1 General formulation of the elasticity problem

Let us consider a three-dimensional body denoted as $\Omega$ with a boundary $S = \partial\Omega$ in the Cartesian coordinate system $Ox_1, x_2, x_3$. The body is fixed on a portion of the boundary $S_u \in S$ where the displacement vector $\overrightarrow{u}(\overrightarrow{x})$ is zero for all points $\overrightarrow{x}$ in $S_u$ [1].

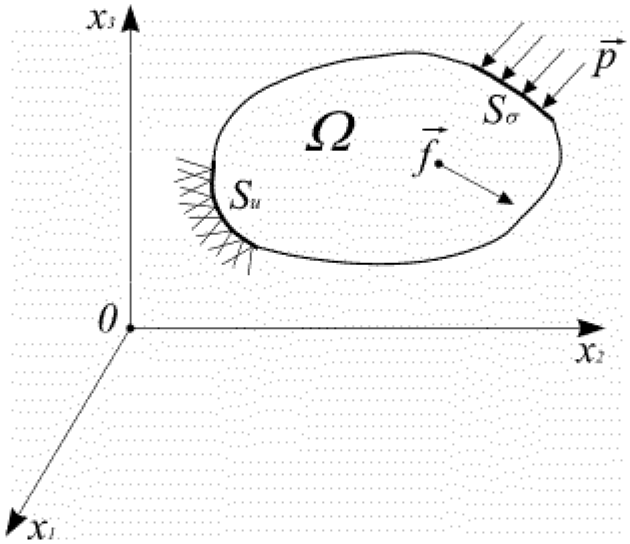


Figure 1

The deformation of the body occurs under the influence of surface forces $\overrightarrow{p}$ and volume forces $\overrightarrow{f}$. The stress-strain state of the body at any point $\overrightarrow{x} \in \Omega$ is characterized by the displacement vector $\overrightarrow{u} = (u_1, u_2, u_3)$, the symmetric tensor of linear strains $\hat{E}$ with components $\varepsilon_{ij}$, and the symmetric stress tensor $\hat{\Sigma}$ with components $\sigma_{ij}$ [1].

The relationship between displacements and strains is described by Cauchy's equations, which express the strains as a function of the displacement gradients [1, 2]:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3 \tag{2.1}$$

Stresses are related to strains by Hooke's law, which considers the elastic constants of the material [1, 2]:

$$\sigma_{ij}(\overrightarrow{u}) = \sum_{k,l=1}^{3} a_{ijkl}\varepsilon_{kl}(\overrightarrow{u}), \quad i,j = 1,2,3 \tag{2.2}$$

where $a_{ijkl}$ are the components of the tensor of elastic constants.
Additionally, the equilibrium equations are satisfied [1]:

$$\sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\sigma_{ik}(\overrightarrow{u})) + f_i = 0, \quad i = 1,2,3 \tag{2.3}$$

If we substitute Eq. (2.1) into Eq. (2.2) and then Eq. (2.2) into Eq. (2.3), we obtain three second-order equations for the displacements $u_1$, $u_2$, and $u_3$. These equations are known as Lamé's equations. Three boundary conditions on the surface $S$ must be added to them. The following main types of boundary conditions are distinguished: kinematic boundary conditions, where the displacements $\overrightarrow{u}(\overrightarrow{x})$ are specified on the surface $S_u \in S$ [1], and static boundary conditions, where the stresses $\overrightarrow{\sigma_n}(\overrightarrow{x})$ are specified on the surface $S_\sigma \in S$ [1]. Using Eq. (2.1) and Eq. (2.2), the conditions for the boundary conditions can be easily rewritten in terms of the displacement vector $\overrightarrow{u}$.

## 2.2 Green's formula for elasticity equations

The Lamé equation can be initially formulated as [3, 4]:

$$A\overrightarrow{u} = \overrightarrow{f} \tag{2.4}$$

where the operator $A$ acts on the vector function $\overrightarrow{u}$, defined as:

$$A\overrightarrow{u} = -\left(\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left[\sum_{k,l=1}^{3} a_{1jkl}\varepsilon_{kl}(\overrightarrow{u})\right], ...\right) = -\left(\sum_{j=1}^{3} \frac{\partial}{\partial x_j}[\sigma_1(\overrightarrow{u})], ...\right)$$

The Green's formula for this equation can be determined by first defining the scalar product of two vector functions $\overrightarrow{u}$ and $\overrightarrow{\nu}$ [5, 6]:

$$(\overrightarrow{u}, \overrightarrow{v}) = \int_{\Omega} \overrightarrow{u} \cdot \overrightarrow{\nu} d\Omega, \quad where \quad \overrightarrow{u}(\overrightarrow{x}) \cdot \overrightarrow{\nu}(\overrightarrow{x}) = \sum_{i=1}^{3} u_i(\overrightarrow{x})\nu_i(\overrightarrow{x})$$

Considering the scalar product $(A\overrightarrow{u}, \overrightarrow{\nu})$, we can express it as:

$$(A\overrightarrow{u}, \overrightarrow{\nu}) = -\int_{n} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u})\nu_i d\Omega$$

We then utilize the following relationship to rewrite the equation:

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u}))\nu_i = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u})\nu_i) + \sum_{i,j=1}^{3} \sigma_{ij}(\overrightarrow{u})\frac{\partial \nu_i}{\partial x_j}$$

Subsequently, the final term can be simplified using the properties of the vortex tensor, yielding:

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u}))\nu_i = \sum_{i,j=1}^{3} \sigma_{ij}(\overrightarrow{u})\varepsilon_{ij}(\overrightarrow{\nu}) - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u}\nu_i))$$

Finally, applying the Gauss-Ostrogradsky formula, we obtain the Green's formula for operator A [7]:

$$(A\overrightarrow{u}, \overrightarrow{\nu}) = \int_\Omega \sum_{i,j=1}^{3} \sigma_{i,j}(\overrightarrow{u})\varepsilon_{ij}(\overrightarrow{\nu})d\Omega - \int_\Omega \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u}))\nu_i d\Omega$$

By applying the Gauss-Ostrogradsky formula, the last term can be written as:

$$-\int_\Omega \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(\sigma_{ij}(\overrightarrow{u}))\nu_i d\Omega = -\int_S \sum_{i,j=1}^{3} \sigma_{i,j}(\overrightarrow{u}) n_j \nu_i dS =$$

$$-\int_S \sum_{i=1}^{3} \sigma_{ni}\nu_i dS = -\int_{\partial\Omega} \overrightarrow{\sigma_n} \cdot \overrightarrow{\nu} dS$$

Define the bilinear form $a(\overrightarrow{u}, \overrightarrow{\nu}) = \int_\Omega \sum_{i=1}^{3} \sigma_{ij}(\overrightarrow{u})\varepsilon_{ij}(\overrightarrow{\nu})d\Omega$. The final expression for Green's formula is then given by:

$$(A\overrightarrow{u}, \overrightarrow{\nu}) = a(\overrightarrow{u}, \overrightarrow{\nu}) - \int_{\partial S} \overrightarrow{\sigma_n}\overrightarrow{\nu} dS \tag{2.5}$$

## 2.3 Variational formulation of the elasticity problem

Consider the following functional:

$$J(\vec{u}) = \frac{1}{2}a(\vec{u}, \vec{u}) - \int_\Omega \vec{f}\vec{u} d\Omega - \int_{S_\sigma} \vec{p_n}\vec{u} dS \tag{2.6}$$

This functional, $J(\vec{u})$, is known as the Lagrangian functional [8]. We seek to find its minimum within the linear space of kinematically valid displacements, denoted $H_0 = \vec{u} \in H : \vec{u}(\vec{x}) = 0, \quad \vec{x} \in S$, where $H$ represents a Hilbert space. The necessary condition for the minimum of this functional is $J'(\vec{u}, \vec{\nu}) = 0, \quad \forall \vec{\nu} \in H_0$. From this, we have:

$$a(\vec{u}, \vec{\nu}) - \int_\Omega \vec{f}\vec{\nu} d\Omega - \int_{S_\sigma} \vec{p_n}\vec{\nu} dS = 0$$

Using the Green's formula (2.5) [9], this can be rewritten as:

$$(A\vec{u}, \vec{\nu}) - \int_{\Omega} \vec{f}\vec{\nu} d\Omega + \int_{\partial\Omega} \vec{\sigma_n}\vec{\nu} dS - \int_{\partial\Omega} \vec{p_n}\vec{\nu} dS = 0$$

which simplifies to

$$\int_{\Omega} (A\vec{u} - \vec{f})\vec{\nu} d\Omega + \int_{\partial\Omega} (\vec{\sigma_n} - \vec{p_n})\vec{\nu} dS = 0$$

Given that $\vec{\nu}$ can be any arbitrary vector, it can be chosen such that $\vec{\nu} = 0$ on $\partial\Omega$. With this choice of $\nu$, we have:

$$\int_{\partial\Omega} (\vec{\sigma_n} - \vec{p_n})\vec{\nu} dS = 0$$

This leads to

$$\int_{\Omega} (A\vec{u} - \vec{f})\vec{\nu} d\Omega = 0$$

which implies that:

$$A\vec{u} - \vec{f} = 0 \tag{2.7}$$

This gives us the original equations. By setting $\vec{\nu} \equiv 0$ on $S \setminus S_\sigma$ and using (2.7), we find that $\vec{\sigma_n} = \vec{p_n}$ on $S_\sigma$. As a result, it is demonstrated that the solution to the original boundary problem is found by minimizing the functional (2.6) within the space of kinematically valid displacements $H_0$ [10].

# 3 Finite element approximation of the axisymmetric problem in elasticity theory

## 3.1 Mathematical formulation of the axisymmetric elasticity problem

Let us consider the problem of the stress-strain state of an axisymmetric elastic body. The plane of deformation is denoted as $O_{zr}$. The displacements

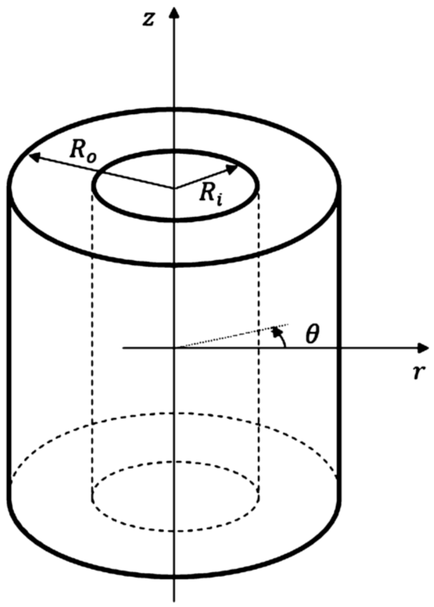


Figure 2

along the $O_r$, $O_z$, and $O_\phi$ axes are represented as $u$, $v$, and $w$ respectively. We assume that the displacements depend only on $r$ and $z$, resulting in $u = u(r, z)$, $v = v(r, z)$, and $w = 0$ [3, 1].

From the Cauchy relations (2.1) for the components of the strain tensor, the following relations can be obtained:

$$\begin{aligned} &\varepsilon_{rr} = \frac{\partial u_r}{\partial z}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{\phi\phi} = \frac{u}{r}, \quad \gamma_{rz} = 2\varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \\ &\gamma_{z\phi} = 2\varepsilon_{z\phi} = 0, \quad \gamma_{r\phi} = 2\varepsilon_{r\phi} = 0, \quad \sigma_{z\phi} = \sigma_{r\phi} = 0 \end{aligned} \tag{3.1}$$

where $u_r = \dfrac{\partial u}{\partial r}$, $u_z = \dfrac{\partial u}{\partial z}$ [11].

In the case of isotropy, the Hooke's law relations (2.2) can be expressed as:

$$\begin{aligned}
\sigma_{rr} &= D_{11}\varepsilon_{rr} + D_{12}\varepsilon_{zz} + D_{14}\varepsilon_{\phi\phi}, \\
\sigma_{zz} &= D_{12}\varepsilon_{rr} + D_{22}\varepsilon_{zz} + D_{24}\varepsilon_{\phi\phi}, \\
\sigma_{rz} &= D_{33}\gamma_{rz}, \\
\sigma_{\phi r} &= \sigma_{\phi z} = 0, \\
\sigma_{\phi\phi} &= D_{14}\varepsilon_{rr} + B_{24}\varepsilon_{zz} + D_{44}\varepsilon_{\phi\phi}
\end{aligned} \tag{3.2}$$

where $D_{11} = D_{22} = D_{44} = D$, $D_{12} = D_{14} = D_{24} = \dfrac{\nu}{1-\nu}D$, $D_{33} = \dfrac{1-2\nu}{2(1-\nu)}D$, and $D = \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}$ [11, 1].

Introducing the pseudovectors of strains and stresses as $[\varepsilon] = [\varepsilon_{rr}, \varepsilon_{zz}, \varepsilon_{rz}, \varepsilon_{\phi\phi}]^T$ and $[\sigma] = [\sigma_{rr}, \sigma_{zz}, \sigma_{rz}, \sigma_{\phi\phi}]^T$, we can express the relationships (3.1) and (3.2) in matrix form. The Cauchy relationship can be written as:

$$[\varepsilon] = [A]\begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{where} \quad [A] = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 \\ 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial r} \\ \dfrac{1}{r} & 0 \end{bmatrix} \tag{3.3}$$

Furthermore, Hooke's law can be expressed as:

$$[\sigma] = [D][\varepsilon], \quad [D] = D\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 & \dfrac{\nu}{1-\nu} \\ \dfrac{\nu}{1-\nu} & 1 & 0 & \dfrac{\nu}{1-\nu} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 1 \end{bmatrix} \tag{3.4}$$

Assuming the absence of body forces, the vector of external loading is denoted as $[p] = [p_x, p_y]^T$. The Lagrangian functional in matrix notation is written as:

$$J^{(e)} = \frac{1}{2}\int_{\Omega}[u^{(e)}]^T[D][u^{(e)}]d\Omega - \int_{S_\sigma}[p^{(e)}]^T(u^{(e)})dS \tag{3.5}$$

where $u^{(e)}$ and $p^{(e)}$ represent the approximate displacement field and the external load distribution, respectively [11, 1, 12].

## 3.2 Finite Element Scheme Based on Linear and Quadratic Basis Functions

To approximate displacements in equation (3.5), we utilize linear and quadratic basis functions of different orders. The overall domain is represented as the union of these elements:

$$\Omega = \bigcup_{\varepsilon=1}^{N} \Omega^{(\varepsilon)}$$

The values of displacements $u$ and $v$ on the quadrilateral element $\Omega^{(\varepsilon)}$ are approximated using linear functions:

$$u = [N^{(\varepsilon)}]^T[u^{(\varepsilon)}], \quad v = [N^{(\varepsilon)}]^T[v^{(\varepsilon)}]$$

Here $[N^{(\varepsilon)}]$ is the vector of shape functions of size $s$, $[u^{(\varepsilon)}] = [u_1^{(\varepsilon)}, u_2^{(\varepsilon)}, u_3^{(\varepsilon)}, u_4^{(\varepsilon)}]^T$ and $[v^{(\varepsilon)}] = [v_1^{(\varepsilon)}, v_2^{(\varepsilon)}, v_3^{(\varepsilon)}, v_4^{(\varepsilon)}]^T$ are the coefficient vectors for the expansion of displacements on the element with index $\varepsilon$. For linear basis functions, $s = 4$, and the vector of shape functions takes the form:

$$[N^{(\varepsilon)}]^T = [N_1^{(\varepsilon)}(x, y), N_2^{(\varepsilon)}(x, y), N_3^{(\varepsilon)}(x, y), N_4^{(\varepsilon)}(x, y)]$$

where $N_i^{(\varepsilon)}$ are the shape functions [13].

For quadratic basis functions:

$$[N^{(\varepsilon)}]^T = [N_1^{(\varepsilon)}(x,y), N_2^{(\varepsilon)}(x,y), \ldots, N_8^{(\varepsilon)}(x,y)], \quad s = 8$$

We introduce the generalized displacement vector:

$$U^{(\varepsilon)} = [u^{(\varepsilon)}, v^{(\varepsilon)}]^T \tag{3.6}$$

Then, for the strain vector on element $\varepsilon$, we have:

$$[\varepsilon] = [B^{(\varepsilon)}][U^{(\varepsilon)}] \tag{3.7}$$

where the matrix $[B^{(\varepsilon)}]$ has the form:

$$[B^{(e)}] = \begin{bmatrix} \dfrac{\partial N_1^{(e)}}{\partial r} & \cdots & \dfrac{\partial N_s^{(e)}}{\partial r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial N_1^{(e)}}{\partial z} & \cdots & \dfrac{\partial N_s^{(e)}}{\partial z} \\ \dfrac{\partial N_1^{(e)}}{\partial z} & \cdots & \dfrac{\partial N_s^{(e)}}{\partial z} & \dfrac{\partial N_1^{(e)}}{\partial r} & \cdots & \dfrac{\partial N_s^{(e)}}{\partial r} \\ \dfrac{1}{r}N_1 & \cdots & \dfrac{1}{r}N_s & 0 & \cdots & 0 \end{bmatrix}$$

For the subintegral function of the quadratic part of the functional (3.5), we obtain an approximation on element $e$ as:

$$\frac{1}{2}[\varepsilon]^T[D][\varepsilon] \approx \frac{1}{2}[U^{(e)}]^T[K^{(e)}][U^{(e)}] \tag{3.8}$$

where

$$[K^{(e)}] = [B^{(e)}]^T[D][B^{(e)}]$$

After performing the necessary calculations, the matrix $[K^{(e)}]$ is represented as:

$$[K^{(e)}] = \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \tag{3.9}$$

Where the blocks $k_{lm}^{(e)}$ are of size s $\times$ s, and their components are given by the following expressions:

$$\begin{aligned}
k_{11}^{(e)}[i,j] &= (D_{11}\frac{\partial N_j^{(e)}}{\partial r}r + D_{14}N_j^{(e)})\frac{\partial N_i^{(e)}}{\partial r} + \\
&\quad + D_{33}\frac{\partial N_i^{(e)}}{\partial z}\frac{N_j^{(e)}}{\partial z}r + (D_{14}\frac{\partial N_j^{(e)}}{\partial r} + D_{44}\frac{N_j^{(e)}}{r})N_i^{(e)} \\
k_{12}^{(e)}[i,j] &= D_{12}\frac{\partial N_i^{(e)}}{\partial r}\frac{\partial N_j^{(e)}}{\partial z}r + D_{33}\frac{N_i^{(e)}}{\partial z}\frac{N_j^{(e)}}{\partial r}r + D_{24}\frac{\partial N_j^{(e)}}{\partial z}N_i^{(e)} \\
k_{21}^{(e)}[i,j] &= (D_{12}\frac{\partial N_j^{(e)}}{\partial r}r + D_{24}N_j^{(e)})\frac{\partial N_i^{(e)}}{\partial z} + D_{33}\frac{\partial N_i^{(e)}}{\partial r}\frac{N_j^{(e)}}{\partial z}r \\
k_{22}^{(e)}[i,j] &= D_{22}\frac{N_i^{(e)}}{\partial z}\frac{N_j^{(e)}}{\partial z}r + D_{33}\frac{\partial N_i^{(e)}}{\partial r}\frac{N_j^{(e)}}{\partial r}r
\end{aligned}$$

Here, r $^{(e)} = \sum_{i=1}^{s} R_i N_i^{(e)}$.

After integrating over the element for the quadratic part of the functional, we obtain the element-wise approximation:

$$\frac{1}{2}\int_{\Omega^{(e)}} [\varepsilon]^T[D][\varepsilon] \approx \frac{1}{2}[U^{(e)}]^T[K^{(e)}][U^{(e)}] \tag{3.10}$$

where $[K^{(e)}]$ is the stiffness matrix of the element, given by:

$$[K^{(e)}] = 2\pi\int_{\Omega^{(e)}} [k^{(e)}]d\Omega \tag{3.11}$$

Similarly, the integral of the given forces over the part of the boundary $S \cap \Omega^{(e)}$ is approximated as:

$$\int_{S^{(e)}} [p]^T[u]dS \approx [U^{(e)}]^T[F^{(e)}] \tag{3.12}$$

where $[F^{(e)}]$ is the nodal load vector on the element, given by:

$$[F^{(e)}]^T = 2\pi \int_{S\cap\Omega^{(e)}} [p_r^{(e)} N_1^{(e)}, \ldots, p_r^{(e)} N_s^{(e)}, p_z^{(e)} N_1^{(e)}, \ldots, p_z^{(e)} N_s^{(e)}] dS$$

where $p_r^{(e)} = \sum_{i=1}^{s} R_i p_{ri}^{(e)} N_i^{(e)}$ and $p_z^{(e)} = \sum_{i=1}^{s} R_i p_{zi}^{(e)} N_i^{(e)}$, and $p_{ri}^{(e)}, p_{zi}^{(e)}$ are the values of the respective loads at node $i$. Thus, the original functional $J(\mathbf{u})$ is approximated on the element by a quadratic function of $2s$ independent variables:

$$J^{(e)}(\mathbf{U}) \approx \frac{1}{2}[\mathbf{U}^{(e)}]^T[K^{(e)}][\mathbf{U}^{(e)}] - [\mathbf{U}^{(e)}]^T[\mathbf{F}^{(e)}] \tag{3.13}$$

Summing over the elements and combining similar terms, we obtain the overall approximation of the functional:

$$J = \sum_{e=1}^{N} J^{(e)} \approx \frac{1}{2}[\mathbf{U}]^T[K][\mathbf{U}] - [\mathbf{U}]^T[\mathbf{F}] \tag{3.14}$$

where $[\mathbf{U}] = [u_1, v_1, u_2, v_2, \ldots, u_m, v_m]$ is the global vector of displacement coefficients, m is the total number of unknowns, $[K]$ is the global stiffness matrix of size $m \times m$, and $[\mathbf{F}]$ is the global load vector of size $m$. It is evident that the minimization of the function (3.5) reduces to solving a system of linear algebraic equations $[K][\mathbf{U}] = [\mathbf{F}]$.

## 3.3 The algorithm for forming the global stiffness matrix and the global load vector

The global stiffness matrix is composed of blocks with a size of $2 \times 2$, reflecting the two displacements at each node. To construct the global stiffness matrix, the internal structure of the element stiffness matrix is mapped to the external structure, taking into account the different numbering of coefficients in the element and global displacement vectors [13].

Let $(s,t)$ represent the index of a block in the element stiffness matrix, and $(i,j)$ denote the index of a component within the block based on local element unknown numbering. The global indices of unknowns $i$ and $j$ for element $e$ are denoted as $G^{(e)}(i)$ and $G^{(e)}(j)$ respectively. The relationship between the components of the local and global stiffness matrices can be expressed as follows:

$$k_{st}^{(e)}[i,j] \longrightarrow K_{G^{(e)}(i),G^{(e)}(j)}[s,t] \longrightarrow K[((G^{(e)}(i)-1)\times 2)+s,((G^{(e)}(j)-1)\times 2)+t].$$

In this way, the $(i,j)$-th element of the $(s,t)$-th block in the local stiffness matrix corresponds to the $(s,t)$-th element of the $(G^{(e)}(i),G^{(e)}(j))$-th block in the global stiffness matrix.

The global load vector $[F]$ has dimensions of $2\times m$, where $m$ is the number of nodes. Let $s$ represent the index of a block in the element load vector, and $i$ represent the index of a component within the block based on the local node numbering in the element. The global index of the $i$-th unknown for element $e$ is denoted as $G^{(e)}(i)$. The relationship between the components of the local and global load vectors can be expressed as follows:

$$F_s^{(e)}[i] \longrightarrow F_{G^{(e)}(i)}[s] \longrightarrow F[((G^{(e)}(i)-1)\times 2)+s].$$

Thus, the $i$-th element of the $s$-th block in the local load vector corresponds to the $s$-th element of the $G^{(e)}(i)$-th block in the global load vector [14].

# 4 Boundary element approximation of the axisymmetric problem in elasticity theory

## 4.1 Fundamental singular solutions

The solution (2.1)-(2.3) for radial and axial ring loads, given by $2\pi r_0 e_r$ and $2\pi r_0 e_z$, respectively, representing the displacement field (corresponding to radial and axial loads) uniformly distributed along a ring $\{(r_0, \phi, z_0), 0 \leq \phi < 2\pi\}$ with intensities $e_r$ and $e_z$, respectively, can be expressed as:

$$u_i = G_{ij} e_j, \quad i, j = r, z \tag{4.1}$$

where $e_j$ – components of a unit concentrated force applied at point $(r_0, z_0)$:

$$\begin{aligned}
G_{rr} &= \frac{1}{4\pi\mu(1-\nu)} \left[ \frac{(3-4\nu)(r^2+r_0^2)+4(1-\nu)Z^2}{2ra} K(m) - \right. \\
&\quad \left. - \left( \frac{3-4\nu}{2r} a - \frac{(r^2-r_0^2)^2 - Z^4}{4raR^2} \right) E(m) \right] \\
G_{zr} &= \frac{Z}{4\pi\mu(1-\nu)} \left[ \frac{r^2-r_0^2+Z^2}{2aR^2} E(m) - \frac{1}{2a} K(m) \right] \\
G_{rz} &= \frac{r_0 Z}{4\pi\mu(1-\nu)} \left[ \frac{r^2-r_0^2-Z^2}{2aR^2} E(m) + \frac{1}{2ra} K(m) \right] \\
G_{zz} &= \frac{r_0}{4\pi\mu(1-\nu)} \left[ \frac{3-4\nu}{a} K(m) + \frac{Z^2}{aR^2} E(m) \right]
\end{aligned}$$

where $(r, z)$ – observation point

$$\begin{aligned}
Z &= z - z_0, \\
a &= \sqrt{(r+r_0)^2 + Z^2}, \\
R &= \sqrt{(r-r_0)^2 + Z^2},
\end{aligned}$$

$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}, \quad E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m\sin^2\phi}d\phi$ - the complete elliptic integrals of the first and second order with $m = \frac{4rr_0}{a^2}$.

The deformations are determined by substituting (4.1) into (3.1):

$$\varepsilon ij = B_{ijk}\varepsilon_k, \quad k = r, z, \tag{4.2}$$

where $B_{rri} = \frac{\partial G_{ri}}{\partial r}, \quad B_{\phi\phi i} = \frac{G_{ri}}{r}, \quad B_{zzi} = \frac{\partial G_{zi}}{\partial z}, \quad B_{rzi} = \frac{1}{2}\left(\frac{\partial G_{ri}}{\partial z} + \frac{\partial G_{zi}}{\partial r}\right)$

We determine the stresses by substituting (4.2) into (3.2):

$$\sigma_{ij} = T_{ijk}e_k \tag{4.3}$$

where

$$T_{rri} = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial G_{ri}}{\partial r} + \nu\left(\frac{G_{ri}}{r} + \frac{\partial G_{zi}}{\partial z}\right)\right]$$
$$T_{\phi\phi i} = \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial G_{ri}}{\partial r} + (1-\nu)\frac{G_{ri}}{r} + \nu\frac{\partial G_{zi}}{\partial z}\right]$$
$$T_{zzi} = \frac{2\mu}{1-2\nu}\left[\nu\left(\frac{\partial G_{ri}}{\partial r} + \frac{G_{ri}}{r}\right) + (1-\nu)\frac{\partial G_{zi}}{\partial z}\right]$$
$$T_{rzi} = \mu\left(\frac{\partial G_{ri}}{\partial z} + \frac{\partial G_{zi}}{\partial r}\right)$$

Note: In the formulas (4.2), (4.3), we assume that the subscripts next to symbols $\varepsilon, \sigma, B$ and $T$ take only permissible values.

The forces are determined at points on the surface with the external normal vector $\vec{n}$ by substituting (4.3) into (5):

$$F_{ij} = T_{ijk}e_k, \quad i, j = r, z \tag{4.4}$$

where

$$F_{ri} = \frac{2\mu}{1-2\nu}\left((1-\nu)\frac{\partial G_{ri}}{\partial r} + \nu\frac{\partial G_{ri}}{r} + \nu\frac{\partial G_{zi}}{\partial z}\right)n_r + \mu\left(\frac{\partial G_{ri}}{\partial z} + \frac{\partial G_{zi}}{\partial r}\right)n_z$$
$$F_{zi} = \frac{2\mu}{1-2\nu}\left((1-\nu)\frac{\partial G_{zi}}{\partial z} + \nu\frac{\partial G_{ri}}{r} + \nu\frac{\partial G_{ri}}{\partial r}\right)n_z + \mu\left(\frac{\partial G_{ri}}{\partial z} + \frac{\partial G_{zi}}{\partial r}\right)n_r$$

The partial derivatives of the elliptic integrals required for computation are calculated using the formulas:

$$\frac{\partial K}{\partial m} = \frac{E}{2m(1-m)} - \frac{K}{2m}$$
$$\frac{\partial E}{\partial m} = \frac{E-K}{2m}$$

Based on the known results from [5,8], we obtain the singularity behavior: functions $G_{ij}, F_{ij}, B_{ijk}, T_{ijk}$ are singular at $R = 0$. Let's assess the order of singularity, which will be:

$$R^{-1}\ln\left(\frac{1}{R}\right) \quad \text{(weak singularity)} \quad \text{for functions} \quad G_{ij}, F_{rr}, F_{zz}, \quad \text{and}$$
$$R^{-1} \quad \text{(strong singularity)} \quad \text{for functions} \quad F_{rz}, F_{zr}.$$

If $r = 0$, certain functions from the given ones will also be singular. Therefore, points on the specified meridional contour $L$ with zero radial coordinate should be moved a small distance in the radial direction.

## 4.2 Integral equations

Therefore, the solution to the problem (1)-(5) at any internal point of the meridional cross-section is obtained by convolving the fundamental solution (6), (7)-(9) with the so-called fictitious intensities of surface forces $\phi$ along the meridional contour $L$ and volume forces over the meridional cross-section $S$ [3]:

$$u_i(P) = \int_L G_{ij}(P,Q)\phi_j(Q)dL(Q) + \\ + \int_S G_{ij}(P,A)\rho(A)F_i(A)dS(A), \quad i,j = r,z \tag{4.5}$$

$$\varepsilon_{ij}(P) = \int_L B_{ijk}(P,Q)\phi_k(Q)dL(Q) + \\ + \int_S B_{ijk}(P,A)\rho(A)F_i(A)dS(A), \quad i,j = r,z \tag{4.6}$$

$$\sigma_{ij}(P) = \int_L T_{ijk}(P,Q)\phi_k(Q)dL(Q) + \\ + \int_S T_{ijk}(P,A)\rho(A)F_i(A)dS(A), \quad i,j = r,z \tag{4.7}$$

$$t_i(P) = \int_L F_{ij}(P,Q)\phi_j(Q)dL(Q) + \\ + \int_S F_{ij}(P,A)\rho(A)F_i(A)dS(A) \quad i,j = r,z \tag{4.8}$$

In equations (10)-(13), the point $P$ represents the observation point. The normal vector for $F_{ij}(P,Q)$ in (13) is evaluated at point $P$. The direction of traversal of $L$ is chosen counterclockwise for the internal problem and clockwise for the external problem. The integrals over $S$ in equations (10)-(13) do not involve unknown functions. To simplify the presentation without loss of generality, we consider the case where volume forces are absent, i.e. $F_r = F_z = 0$

Based on relations (10) and (13), we construct a numerical algorithm for solving our problem. By directing point $P$ towards the meridional contour in (10) and (13), we obtain:

$$u_i(P) = \int_L G_{ij}(P,Q)\phi_j(Q)dL(Q) \tag{4.9}$$

$$t_i(P) = \int_L F_{ij}(P,Q)\phi_j(Q)dL(Q) \pm \mu_{ij}\phi_j(P) \tag{4.10}$$

If point $P$ is not a corner point (i.e., it has a unique tangent at that point), then we have:

$$\mu_{rr} = \mu_{zz} = \frac{1}{2}, \qquad \mu_{rz} = \mu_{zr} = 0$$

The curvilinear integral integral in (15) is the Cauchy principal value integral. In expression $\mu_{ij}\phi_j(P)$ in (15), we take the sign "+" in the case of an interior problem and the sign "-" in the case of an exterior problem.

Equations (14) and (15) are boundary integral equations corresponding to the problem (1)-(5).

Sometimes it is convenient, especially in the case of a curvilinear boundary of a meridional cross-section, to express displacements and forces in terms of their normal and tangential components. For displacements and forces, we have:

$$u_n = u_r n_r + u_z n_z, \quad u_\tau = u_r(-n_z) + u_z n_r \tag{4.11}$$

$$t_n = t_r n_r + t_z n_z, \quad t_\tau = t_r(-n_z) + t_z n_r \tag{4.12}$$

By substituting (14) into (16) and (15) into (17), we obtain the boundary integral equations for the normal and tangential components:

$$u_i(P) = \int_L G_{ij}(P,Q)\phi_j(Q)dL(Q), \quad (18) \tag{4.13}$$

$$t_i(P) = \int_L F_{ij}(P,Q)\phi_j(Q)dL(Q) \pm \mu_{ij}\phi_j(Q)dL(Q), \quad i = n, \tau \tag{4.14}$$

where the functions $G_{ij}$, $F_{ij}$, $\mu_{ij}$, $i = n, \tau$, can be determined easily:

$$G_{nj} = G_{rj}n_r + G_{zj}n_z, \quad G_{\tau j} = -G_{rj}n_z + G_{zj}n_r$$

$$F_{nj} = F_{rj}n_r + F_{zj}n_z, \quad F_{\tau j} = -F_{rj}n_z + F_{zj}n_r$$

$$\mu_{nj} = \mu_{rj}n_r + \mu_{zj}n_z, \quad \mu_{\tau j} = -\mu_{rj}n_z + \mu_{zj}n_r$$

## 4.3 Numerical scheme for solving integral equations

### Bubnov-Galerkin Method

Based on the obtained data from the study [9,7], we can apply the following approach: we divide the meridional contour $L$ into curves $Lp \quad p = 1, ..., N$ which are called boundary elements (BE). We assume that irregular points on $L$ (corner points, points of abrupt load change, points of boundary condition change) cannot be internal points of BE.

Equations (14) and (15), (18) and (19) are transformed into the following form:

$$u_j(P) = \sum_{q=1}^{N} \sum_{j=r,z} \int_{L_q} G_{ij}(P, Q)\phi_j(Q)dL_q(Q) \tag{4.15}$$

$$t_j(P) = \sum_{q=1}^{N} \sum_{j=r,z} \int_{L_q} F_{ij}(P, Q)\phi_j(Q)dL_q(Q) + \sum_{j=r,z} \mu_{ij}\phi_j(P), \quad i = r, z \tag{4.16}$$

On the $p$-th BE, we select $s > 1$ nodal points $pkx, k = 1, ..., s$, which are numbered in the direction of $L$ traversal, and we choose them in such a way that $x_j^{p,s} = x_j^{p+1,1}$. Here and further, we assume that $x^{N+1,1}j = x_j^{1,1}$.

The curves $L_p$ and $L$ are approximated using parametrically defined curves $\widetilde{L_p}$ and $\widetilde{Lp}$ respectively:

$$\widetilde{L_p} = \left\{ \xi^p(t) | \xi_j^p(t) = \sum_{k=1}^{S} x_j^{p,k} N^k(t), \quad t \in [-1, 1] \right\}, \quad \widetilde{L} = \bigcup_{p=1}^{N} \widetilde{L_p} \tag{4.17}$$

where $N(t)k$, $k = 1, \ldots, s$, are Lagrange polynomials of degree $(s - 1)$ defined on the interval $[-1, 1]$. The element $d\widetilde{L_p}(\xi^p(t))$ of the curve $\widetilde{L_p}$ is determined by the formula:

$$d\widetilde{L_p}(t) = \sqrt{\sum_{j=r,z} \xi_j^{p'}(t)\xi_j^{p'}(t)dt} = J_p(t)dt \tag{4.18}$$

$$\xi_j^{p'}(t) = \sum_{k=1}^{S} x_j^{p,k} N^{k'}(t)$$

where $\xi_k$ are the coordinates of the Lagrange polynomial of degree $(s-1)$, $N(t)$, defined on the interval $[-1, 1]$.

The coordinates of the outward normal to the curve $\widetilde{L_p}$ are computed using the formulas:

$$n_r(\xi^p(t)) = \frac{\xi_z^{p'}(t)}{J_p(t)}, \quad n_z(\xi^p(t)) = -\frac{\xi_r^{p'}(t)}{J_p(t)}$$

The unknown function $\phi_j$ is approximated on $\widetilde{L}$ by the function:

$$\widetilde{\phi}_j(\xi^p(t)) = \sum_{q=1}^{N} \sum_{k=1}^{S} \phi_j^{q,k} N^k(t) \delta_{pq} = \sum_{k=1}^{S} \phi_j^{p,k} N^k(t) \tag{4.19}$$

where $\phi_j^{p,k} = \phi_j(x^{p,k}), \quad \delta_{pq} = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$

If the last point $x^{p,s}$ of the BE is regular, then $\phi_j^{p,s} = \phi_j^{p+1,1}$.

Here and further, we assume that $\phi_j^{N+1,1} = \phi_j^{1,1}$.

Taking into account (20)-(24), we present the expressions for the residuals:

$$r_i^u(x) = \sum_{q=1}^{N} \sum_{k=1}^{S} \sum_{j=r,z} \int_{-1}^{1} G_{ij}(x, \xi^q(t)) \phi_j^{q,k} N^k(t) J_q(t) dt - u_i(x) \tag{4.20}$$

$$\begin{aligned} r_i^t(x) = \sum_{q=1}^{N} \sum_{k=1}^{S} \sum_{j=r,z} \int_{-1}^{1} F_{ij}(x, \xi^q(t)) \phi_j^{q,k} N^k(t) J_q(t) dt - u_i(x) - \\ - t_j(x) \pm \sum_{j=r,z} \mu_{ij} \phi_j(x) \end{aligned} \tag{4.21}$$

Applying the Bubnov-Galerkin procedure [10] to (26) and (27), we obtain:

$$\sum_{q=1}^{N}\sum_{k=1}^{S}\sum_{j=r,z}\phi_j^{q,k}\int_{-1}^{1}G_{ij}(x^p(\eta),\xi^q(t))N^k(t)J_q(t)dtJ_p(\eta)d\eta- \\ -\sum_{k=1}^{S}u_i^{p,k}\int_{-1}^{1}N^l(\eta)N^k(\eta)J_p(\eta)d\eta=0 \tag{4.22}$$

$$\sum_{q=1}^{N}\sum_{k=1}^{S}\sum_{j=r,z}\phi_j^{q,k}\int_{-1}^{1}F_{ij}(x^p(\eta),\xi^q(t))N^k(t)J_q(t)dtJ_p(\eta)d\eta\pm \\ \pm\sum_{k=1}^{S}\sum_{j=r,z}\phi_j^{p,k}\int_{-1}^{1}N^l(\eta)\mu_{ij}N^k(\eta)J_p(\eta)d\eta+ \\ -\sum_{k=1}^{S}t_i^{p,k}\int_{-1}^{1}N^l(\eta)N^k(\eta)J_p(\eta)d\eta=0 \tag{4.23}$$

where $l=1,\cdots,S$, and $u_i^{p,k}=u_i(x^{p,k}),\quad t_i^{p,k}=t_i(x^{p,k})$

**Calculation of the coefficients of the SLAE**

From all equations (28) and (29) written for the node $x^{p,l}$, we select only the equations corresponding to the specified boundary conditions on the $p$-th BE. For the entire set of boundary values on all BEs, we obtain a system of linear algebraic equations (SLAE) $A\phi=b$ of size $2sN$. We apply unknowns reduction by connecting two BEs for all regular nodes. We combine certain similar equations assigned to nodes in this set. We replace these equations with the newly formed ones. Similarly, we apply this procedure for the columns.

Taking into account (25), we obtain the final system of linear algebraic equations:

$$A_{\text{fin}}\phi_{\text{fin}}=b_{\text{fin}}$$

of dimension $2sN-2\cdot(\text{N - number of irregular points})$. When the unknowns $\phi_j^{q,k}$ are found, the discrete representation of the integral equations (11) and

(12) is given by:

$$\varepsilon_{im}(x) = \sum_{q=1}^{N}\sum_{k=1}^{s}\sum_{j=r,z} \phi_j^{q,k} \int_{-1}^{1} B_{imj}(x, \xi^q(t)) N^k(t) J_q(t) dt$$

$$\sigma_{im}(x) = \sum_{q=1}^{N}\sum_{k=1}^{s}\sum_{j=r,z} \phi_j^{q,k} \int_{j}^{1} T_{imj}(x, \xi^q(t)) N^k(t) J_q(t) dt$$

what we use to compute $\varepsilon_{im}$ and $\sigma_{im}$ at any internal point of the meridional cross-section. The quality of the approach, i.e., its efficiency, depends on the efficiency of solving the system of linear algebraic equations. Based on these assumptions, it is now worth describing the algorithms used to calculate various types of integrals.

To compute integrals of the form:

$$\int_{-1}^{1} N^l(\eta) N^k(\eta) J_p(\eta) d\eta \tag{4.24}$$

and the external integrals in expressions of the form:

$$\begin{gathered} \int_{-1}^{1} N^l(\eta) \int_{-1}^{1} G_{ij}(x^p(\eta), \xi^q(t)) N^k(t) J_q(t) dt J_p(\eta) d\eta, \\ \text{and} \\ \int_{-1}^{1} N^l(\eta) \int_{-1}^{1} F_{ij}(x^p(\eta), \xi^q(t)) N^k(t) J_q(t) dt J_p(\eta) d\eta \end{gathered} \tag{4.25}$$

we use Gaussian quadrature formulas.

For the computation of internal integrals in expressions of the form (31) in the case $p \neq q$, we also apply Gaussian formulas. In the case where $p = q$, in the corresponding (coincident) points $x^p(t)$ and $\xi^q(\eta)$, the kernels $G_{ij}$ and $F_{ij}$ have singularities of order $\ln \frac{1}{R}$ or $\frac{1}{R}$. In this case, we either use Gaussian formulas that do not have coincident nodes, or we employ special approaches to compute the internal integrals.

# 5 Combined adaptation method

## 5.1 Error Estimation and Relative Error Analysis

In the study by Dyyak et al. [15], a combined algorithm incorporating domain decomposition and $h$-adaptation is proposed for solving contact problems in elasticity. The algorithm aims to enhance the accuracy and efficiency of the solution by integrating various techniques.

The researchers introduce an error estimation approach to compare the precision of stresses obtained from the Finite Element Method (FEM) and the Boundary Element Method (BEM). This approach involves the calculation of the effective stress, denoted as $\sigma_{\text{eff}}$, given by:

$$\sigma_{\text{eff}} = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2},$$

assuming the validity of the plain strain theory. The component $\sigma_{33}$ can be expressed as:

$$\sigma_{33} = \frac{\lambda}{2(\lambda + \mu)}(\sigma_{11} + \sigma_{22}),$$

where $\lambda$ and $\mu$ are material constants.

To compare stresses at different points while considering all tensor components equally, the authors propose the effective stress difference, $\Delta\sigma_{FB}(x)$, computed as:

$$\Delta\sigma_{FB}(x) = \sqrt{\sum_{i,j=1}^{3} (\sigma_{Fij} - \sigma_{Bij})^2}.$$

Simplifying the formula for the case where $\sigma_{13} = \sigma_{23} = 0$, we have:

$$\Delta\sigma_{FB}(x) = \sqrt{\sum_{i,j=1}^{3} (\sigma_{Fij} - \sigma_{Bij})^2}.$$

In the context of mesh adaptivity, the researchers detect finite elements $\Omega_e$ that require refinement. For each $\Omega_e$, the mean root square of the effective stress difference is computed using the $L_2$-norm over the finite element:

$$\frac{\Delta\sigma_{FB}(\Omega_e)}{\|\Omega_e\|} = \frac{\sqrt{\int_{\Omega_e} \Delta\sigma_{FB}^2 d\Omega_e}}{\sqrt{\int_{\Omega_e} 1 d\Omega_e}}.$$

Similarly, the root mean square of the effective stress obtained from BEM in $\Omega_e$ is given by:

$$\overline{\sigma_{B\Omega_e}} = \frac{\sqrt{\int_{\Omega_e} \sum_{i,j=1}^{3} \sigma_{Bij}^2 d\Omega_e}}{\sqrt{\int_{\Omega_e} 1 d\Omega_e}}.$$

The authors propose an estimation of the relative error for FEM stresses using the formula:

$$\eta = \frac{\overline{\Delta\sigma_{FB}(\Omega_e)}}{\overline{\sigma_{B\Omega}}} = \frac{\|\Delta\sigma_{FB}\|_{\Omega_e} / \|\Omega_e\|}{\|\sigma_B\|_{\Omega} / \|\Omega\|}.$$

This elementwise estimator can be used as a criterion for adaptivity.

Furthermore, the researchers introduce an effectivity index, denoted as $\theta$, which compares the estimated error with the actual error:

$$\theta = \frac{\overline{\Delta\sigma_{FB}(\Omega_e)}}{\overline{\sigma_{B\Omega}}} / \frac{\overline{\Delta\sigma_{FT}}}{\overline{\sigma_{T\Omega}}},$$

where $\overline{\Delta\sigma_{FB}(\Omega_e)}$ and $\overline{\Delta\sigma_{FT}}$ are obtained from $\sigma_S$ and $\sigma_T$ in a similar manner.

To address the challenge of connecting grids during adaptive mesh refinement, the authors propose a method that constructs the BEM grid based on the FEM grid. Specifically, the FEM grid nodes belonging to the boundary $\Gamma$ are considered as BEM nodes, enabling a direct comparison between the results obtained from the two methods.

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